A generalization of the Sears–Slater transformation and elliptic Lagrange interpolation of type BC_n

Masahiko Ito* and Masatoshi Noumi†

Abstract

The connection formula for the Jackson integral of type BC_n is obtained in the form of a Sears–Slater type expansion of a bilateral multiple basic hypergeometric series as a linear combination of several specific bilateral multiple series. The coefficients of this expansion are expressed by certain elliptic Lagrange interpolation functions. Analyzing basic properties of the elliptic Lagrange interpolation functions, an explicit determinant formula is provided for a fundamental solution matrix of the associated system of q-difference equations.

Keywords: Basic hypergeometric series, Sears–Slater transformation, elliptic Lagrange interpolation
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1 Introduction

Throughout this paper we fix the base $q \in \mathbb{C}$ with $0 < |q| < 1$, and use the notation of q-shifted factorials

$$(u)_\infty = \prod_{l=0}^{\infty}(1-uq^l), \quad (a_1,\ldots,a_r)_\infty = (a_1)_\infty \cdots (a_r)_\infty,$$
$$(u)_\nu = (u)_\infty/(uq^\nu)_\infty, \quad (a_1,\ldots,a_r)_\nu = (a_1)_\nu \cdots (a_r)_\nu \quad (\nu \in \mathbb{Z}).$$

For generic complex parameters $a_1,\ldots,a_r$ and $b_1,\ldots,b_r$, the bilateral basic hypergeometric series ${}_r\psi_r$ is defined by

$${}_r\psi_r\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_r \end{matrix}; q, x\right] = \sum_{\nu=-\infty}^{\infty} \frac{(a_1,\ldots,a_r)_\nu}{(b_1,\ldots,b_r)_\nu} x^\nu \qquad (|b_1\cdots b_r/a_1\cdots a_r| < |x| < 1).$$

There is a celebrated summation formula for very well-poised balanced ${}_6\psi_6$ series

$${}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e} \end{matrix}; q, \frac{a^2q}{bcde}\right] = \frac{(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, \frac{q}{a})_\infty}{(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{a^2q}{bcde})_\infty} \tag{1.1}$$

*Corresponding Author at: School of Science and Technology for Future Life, Tokyo Denki University, Tokyo 120-8551, Japan, E-mail: mito@cck.dendai.ac.jp
†Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan, E-mail: noumi@math.kobe-u.ac.jp

for $|a^2q/bcde|<1$, called *Bailey's sum* (see [4, (5.3.1), p. 140]). On the other hand, there are several transformation formulas for well-poised or very well-poised ${}_{2r}\psi_{2r}$ series due to Sears [14, 15] and Slater [16, 17]. A typical example is

$$
\begin{aligned}
&{}_{2r}\psi_{2r}\left[\begin{matrix} q\sqrt{a},\ -q\sqrt{a},\ b_3,\ \dots,\ b_{2r} \\ \sqrt{a},\ \ -\sqrt{a},\ \frac{aq}{b_3},\ \dots,\ \frac{aq}{b_{2r}} \end{matrix};q,\frac{a^{r-1}q^{r-2}}{b_3\dots b_{2r}}\right] \\
&= \frac{(a_4,\dots,a_r,\frac{q}{a_4},\dots,\frac{q}{a_r},\frac{a_4}{a},\dots,\frac{a_r}{a},\frac{aq}{a_4},\dots,\frac{aq}{a_r},\frac{a_3q}{b_3},\dots,\frac{a_3q}{b_{2r}},\frac{aq}{a_3b_3},\dots,\frac{aq}{a_3b_{2r}},aq,\frac{q}{a})_\infty}{(\frac{q}{b_3},\dots,\frac{q}{b_{2r}},\frac{aq}{b_3},\dots,\frac{aq}{b_{2r}},\frac{a_4}{a_3},\dots,\frac{a_r}{a_3},\frac{a_3q}{a_4},\dots,\frac{a_3q}{a_r},\frac{a_3a_4}{a},\dots,\frac{a_3a_r}{a},\frac{aq}{a_3a_4},\dots,\frac{aq}{a_3a_r},\frac{a_3^2q}{a},\frac{aq}{a_3^2})_\infty} \\
&\quad\times {}_{2r}\psi_{2r}\left[\begin{matrix} \frac{qa_3}{\sqrt{a}},\ -\frac{qa_3}{\sqrt{a}},\ \frac{a_3b_3}{a},\ \dots,\ \frac{a_3b_{2r}}{a} \\ \frac{a_3}{\sqrt{a}},\ -\frac{a_3}{\sqrt{a}},\ \frac{a_3q}{b_3},\ \dots,\ \frac{a_3q}{b_{2r}} \end{matrix};q,\frac{a^{r-1}q^{r-2}}{b_3\dots b_{2r}}\right] \\
&\quad+\mathrm{idem}(a_3;a_4,\dots,a_r)
\end{aligned}
\tag{1.2}
$$

for $|a^{r-1}q^{r-2}/b_3\cdots b_{2r}|<1$, which is called *Slater's transformation formula for a very well-poised balanced ${}_{2r}\psi_{2r}$ series* (see [4, (5.5.2), p. 143]). Here the symbol "$\mathrm{idem}(a_3;a_4,\dots,a_r)$" stands for the sum of the $r-3$ expressions obtained from the preceding one by interchanging a_3 with each a_k $(k=4,\dots,r)$.

The BC_n Jackson integrals, which we are going to discuss below, are a multiple sum generalization of the very well-poised ${}_{2r}\psi_{2r}$ series. They provide with a natural framework for summation/transformation formulas for basic hypergeometric series, from the viewpoint of the Weyl group symmetry and the q-difference equations. The holonomic system of q-difference equations satisfied by a BC_n Jackson integral has been investigated in [3], from which we recall some terminology.

For a function $\varphi=\varphi(z)$ of $z=(z_1,\dots,z_n)\in(\mathbb{C}^*)^n$, we denote by

$$\langle\varphi,z\rangle=\int_0^{z\infty}\varphi(w)\Phi(w)\Delta(w)\frac{d_qw_1}{w_1}\wedge\cdots\wedge\frac{d_qw_n}{w_n}=(1-q)^n\sum_{\nu\in\mathbb{Z}^n}\varphi(zq^\nu)\Phi(zq^\nu)\Delta(zq^\nu),$$

the Jackson integral associated with the multiplicative lattice $zq^\nu=(z_1q^{\nu_1},\dots,z_nq^{\nu_n})\in(\mathbb{C}^*)^n$ $(\nu=(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n)$. In this definition, we specify the integrand by the weight function

$$\Phi(z)=\prod_{i=1}^n\prod_{m=1}^{2s+2}z_i^{\frac12-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty}\prod_{1\le j<k\le n}z_j^{1-2\tau}\frac{(qt^{-1}z_j/z_k)_\infty(qt^{-1}z_jz_k)_\infty}{(tz_j/z_k)_\infty(tz_jz_k)_\infty},$$

where $q^{\alpha_m}=a_m$ and $q^\tau=t$, and the Weyl denominator of type C_n

$$\Delta(z)=\prod_{i=1}^n\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_jz_k)}{z_j}.$$

We call the sum $\langle\varphi,z\rangle$ the *Jackson integral of type BC_n* if it converges. We denote by $W_n=\{\pm1\}^n\rtimes\mathfrak{S}_n$ the Weyl group of type C_n (hyperoctahedral group of degree n); this group acts on the field of meromorphic functions on $(\mathbb{C}^*)^n$ through the permutations and the inversions of variables $z_1,\dots,z_n$. Setting

$$\langle\!\langle\varphi,z\rangle\!\rangle=\frac{\langle\varphi,z\rangle}{\Theta(z)},\quad \Theta(z)=\prod_{i=1}^n\frac{z_i^s\theta(z_i^2)}{\prod_{m=1}^{2s+2}z_i^{\alpha_m}\theta(a_mz_i)}\prod_{1\le j<k\le n}\frac{\theta(z_j/z_k)\theta(z_jz_k)}{z_j^{2\tau}\theta(tz_j/z_k)\theta(tz_jz_k)},$$

where $\theta(u)=(u)_\infty(qu^{-1})_\infty$, we call $\langle\!\langle \varphi, z\rangle\!\rangle$ the *regularized Jackson integral of type* BC_n. We remark that, for any W_n-invariant holomorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$, the regularization $\langle\!\langle \varphi, z\rangle\!\rangle$ is also holomorphic and W_n-invariant as a function of $z\in(\mathbb{C}^*)^n$ (see [3, Definition 3.8]). This regularized Jackson integral $\langle\!\langle \varphi, z\rangle\!\rangle$ is the main object of this paper.

In what follows, we set

$$B=B_{s,n}=\{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{Z}^n\,;\, s-1\ge\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge 0\},$$

so that $|B_{s,n}|=\binom{s+n-1}{n}$. We also use the symbol $\preceq$ for the lexicographic order of $B_{s,n}$. Namely, for $\lambda,\mu\in B_{s,n}$, we denote $\lambda\prec\mu$ if there exists $k\in\{1,2,\dots,n\}$ such that $\lambda_1=\mu_1,\lambda_2=\mu_2,\dots,\lambda_{k-1}=\mu_{k-1}$ and $\lambda_k<\mu_k$. For each $\lambda\in B_{s,n}$, we denote by $\chi_\lambda(z)$ the *symplectic Schur function*

$$\chi_\lambda(z)=\frac{\det\big(z_i^{\lambda_j+n-j+1}-z_i^{-\lambda_j-(n-j+1)}\big)_{1\le i,j\le n}}{\det\big(z_i^{n-j+1}-z_i^{-(n-j+1)}\big)_{1\le i,j\le n}}=\frac{\det\big(z_i^{\lambda_j+n-j+1}-z_i^{-\lambda_j-(n-j+1)}\big)_{1\le i,j\le n}}{\Delta(z)},$$

which is a W_n-invariant Laurent polynomial. With the basis $\{\chi_\lambda(z)\,;\,\lambda\in B_{s,n}\}$, one can construct a holonomic system of q-difference equations of rank $\binom{s+n-1}{n}$ for the Jackson integral of type BC_n.

Proposition 1.1 (q-Difference system [1, 2]) *Let $\boldsymbol{v}(z)$ be row vector of functions in $a_1,\dots,a_{2s+2}$ and t specified by*

$$\boldsymbol{v}(z)=\big(\langle\!\langle \chi_\lambda, z\rangle\!\rangle\big)_{\lambda\in B},$$

where the indices $\lambda\in B_{s,n}$ are arranged in the increasing order by $\preceq$. If the parameters $a_1,a_2,\dots,a_{2s+2}$ and t are generic, there exist invertible $\binom{s+n-1}{n}\times\binom{s+n-1}{n}$ matrices $Y_{a_i}(i=1,2,\dots,2s+2)$ and Y_t whose entries are rational functions of $a_1,a_2,\dots,a_{2s+2}$ and t, not depending on z, such that

$$T_{a_i}\boldsymbol{v}(z)=\boldsymbol{v}(z)\,Y_{a_i},\qquad T_t\,\boldsymbol{v}(z)=\boldsymbol{v}(z)\,Y_t, \tag{1.3}$$

where T_u stands for the q-shift operator in u.

To construct independent solutions of the above system, we define $\binom{s+n-1}{n}$ specific points in $(\mathbb{C}^*)^n$ as follows. Setting

$$Z=Z_{s,n}=\{\mu=(\mu_1,\mu_2,\dots,\mu_s)\in\mathbb{N}^s\,;\,\mu_1+\mu_2+\cdots+\mu_s=n\}$$

so that $|Z_{s,n}|=\binom{s+n-1}{n}$, we denote the lexicographic order of $Z_{s,n}$ by $\preceq$. For an arbitrary $x=(x_1,x_2,\dots,x_s)\in(\mathbb{C}^*)^s$, we consider the points x_μ $(\mu\in Z_{s,n})$ in $(\mathbb{C}^*)^n$ specified by

$$x_\mu=(\underbrace{x_1,x_1t,\dots,x_1t^{\mu_1-1}}_{\mu_1},\underbrace{x_2,x_2t,\dots,x_2t^{\mu_2-1}}_{\mu_2},\dots,\underbrace{x_s,x_st,\dots,x_st^{\mu_s-1}}_{\mu_s})\in(\mathbb{C}^*)^n. \tag{1.4}$$

In order to confirm that the solutions $\boldsymbol{v}(x_\mu)$ $(\mu\in Z_{s,n})$ of the system (1.3) are linearly independent, we need to verify that the determinant of the matrix $\big(\langle\!\langle \chi_\lambda, x_\mu\rangle\!\rangle\big)_{\lambda\in B,\mu\in Z}$ ("Wronskian" of the system (1.3)) is nonzero under the genericity condition for parameters. One of our main results is the following.

Theorem 1.2 (Determinant formula) *The determinant of the matrix* $\big(\langle\langle \chi_\lambda, x_\mu\rangle\rangle\big)_{\lambda\in B,\mu\in Z}$ *is represented explicitly as*

$$\det\Big(\langle\langle \chi_\lambda, x_\mu\rangle\rangle\Big)_{\substack{\lambda\in B\\ \mu\in Z}} = \prod_{k=1}^{n}\left[\left((1-q)\frac{(q)_\infty (qt^{-(n-k+1)})_\infty}{(qt^{-1})_\infty}\right)^{s}\frac{\prod_{1\le i<j\le 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1})_\infty}\right]^{\binom{s+k-2}{k-1}}$$
$$\times\prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\le i<j\le s}\frac{\theta(t^{2r-(n-k)}x_i x_j^{-1})\theta(t^{n-k}x_i x_j)}{t^r x_i}\right]^{\binom{s+k-3}{k-1}}, \tag{1.5}$$

where the rows $\lambda\in B$ *and the columns* $\mu\in Z$ *of the matrix are arranged by* $\preceq$, *respectively.*

Notice that our explicit formula (1.5) of the determinant splits into two parts. The first part is independent of the choice of cycles of the integral (i.e., independent of x), while the second is a function of x only. From this fact we immediately see that the determinant does not vanish if x is generic.

We remark that in the case $s=1$ the matrix size of $(\langle\langle \chi_\lambda, x_\mu\rangle\rangle)_{\lambda,\mu}$ reduces to 1 and (1.5) in Theorem 1.2 becomes the following formula first proved by van Diejen [18]:

$$\langle\langle 1, z\rangle\rangle = \prod_{k=1}^{n}(1-q)\frac{(q)_\infty (qt^{-k})_\infty}{(qt^{-1})_\infty}\frac{\prod_{1\le i<j\le 4}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty}, \tag{1.6}$$

which is equivalent to the q-Macdonald–Morris identity of type $(C_n^\vee, C_n)$ studied by Gustafson [5]. (See [6] for the derivation of (1.6) along the context of this paper. See also [7, 12] for the other derivations.) In this case the last factor including theta functions in (1.5) disappears. Since (1.6) coincides with (1.1) if $n=1$, we can regard (1.5) as a further extension of Bailey's ${}_6\psi_6$ summation theorem.

Next we consider the connection problem among the independent cycles. From a property of functions written by integral representation, the holonomic system of q-difference equations satisfied by $\langle\langle \varphi, z\rangle\rangle$ does not depend on the choice of cycles, i.e. the choice of points $z\in(\mathbb{C}^*)^n$. It also turns out that $\langle\langle \varphi, z\rangle\rangle$ for any $z\in(\mathbb{C}^*)^n$ is expressed as a linear combination of the special solutions $\langle\langle \varphi, x_\mu\rangle\rangle$, $\mu\in Z_{s,n}$, of the system, i.e.,

$$\langle\langle \varphi, z\rangle\rangle = \sum_{\mu\in Z} c_\mu \langle\langle \varphi, x_\mu\rangle\rangle, \tag{1.7}$$

where c_μ are some connection coefficients. We introduce some terminology before we state the explicit form of the connection coefficients c_μ.

Let $\mathcal{O}((\mathbb{C}^*)^n)$ be the $\mathbb{C}$-vector space of holomorphic functions on $(\mathbb{C}^*)^n$. We consider the $\mathbb{C}$-linear subspace $H_{m,n}\subset\mathcal{O}((\mathbb{C}^*)^n$ consisting of all W_n-invariant holomorphic functions $f(z)$ such that $T_{z_i}f(z)=f(z)/(qz_i^2)^m$ $(i=1,\ldots,n)$, where T_{z_i} stands for the q-shift operator in z_i:

$$H_{m,n}=\{f(z)\in\mathcal{O}((\mathbb{C}^*)^n)^{W_n}\,;\,T_{z_i}f(z)=(qz_i^2)^{-m}f(z)\ \ (i=1,\ldots,n)\}. \tag{1.8}$$

The dimension of $H_{m,n}$ as a $\mathbb{C}$-vector space is known to be $\binom{m+n}{n}$. (See [8, Lemma 3.2].)

Theorem 1.3 *For generic* $x\in(\mathbb{C}^*)^s$ *there exists a unique basis* $\{f_\lambda(z)\,;\,\lambda\in Z_{s,n}\}$ *of* $H_{s-1,n}$ *such that*

$$f_\lambda(x_\mu)=\delta_{\lambda\mu}\qquad(\lambda,\mu\in Z_{s,n}), \tag{1.9}$$

where $\delta_{\lambda\mu}$ *is the Kronecker delta. We denote the function* $f_\lambda(z)$ *by the symbol* $E_\lambda(x;z)$.

We call $E_\lambda(x;z)$ the *elliptic Lagrange interpolation functions of type BC_n*. An explicit construction of these functions $E_\lambda(x;z)$ will be given In Section 2. In particular we will prove

Theorem 1.4 *The elliptic Lagrange interpolation functions $E_\lambda(x;z)$ of type BC_n are represented explicitly as*

$$E_\lambda(x;z)=\sum_{\substack{K_1\sqcup\cdots\sqcup K_s\\=\{1,2,\ldots,n\}}}\prod_{i=1}^{s}\prod_{k\in K_i}\prod_{\substack{1\le j\le s\\ j\ne i}}\frac{\theta(x_jt^{\lambda_j^{(k-1)}}z_k)\theta(x_jt^{\lambda_j^{(k-1)}}z_k^{-1})}{\theta(x_jt^{\lambda_j^{(k-1)}}x_it^{\lambda_i^{(k-1)}})\theta(x_jt^{\lambda_j^{(k-1)}}x_i^{-1}t^{-\lambda_i^{(k-1)}})},\tag{1.10}$$

where $\lambda_i^{(k)}=|K_i\cap\{1,2,\ldots,k\}|$, and the summation is taken over all partitions $K_1\sqcup\cdots\sqcup K_s=\{1,2,\ldots,n\}$ such that $|K_i|=\lambda_i$ $(i=1,2,\ldots,s)$.

Here we mention some special cases of Theorem 1.4.

Example 1. The case $n=1$. We have $Z_{s,1}=\{\epsilon_1,\epsilon_2,\ldots,\epsilon_s\}$, where $\epsilon_i=(0,\ldots,0,\overset{i}{\smile}{1},0,\ldots,0)$, and $x_{\epsilon_i}=x_i$, i.e., we obtain

$$E_{\epsilon_i}(x;z)=\prod_{\substack{1\le j\le s\\ j\ne i}}\frac{\theta(x_jz)\theta(x_j/z)}{\theta(x_ix_j)\theta(x_j/x_i)}.\tag{1.11}$$

Example 2. The case $s=2$. We have $Z_{2,n}=\{(r,n-r)\,;\,r=0,1,\ldots,n\}$. Then

$$\begin{aligned}E_{(r,n-r)}(x_1,x_2;z) \quad = \quad &\sum_{\substack{1\le i_1<\cdots<i_r\le n\\ 1\le j_1<\cdots<j_{n-r}\le n}}\prod_{k=1}^{r}\frac{\theta(x_2t^{i_k-k}z_{i_k})\theta(x_2t^{i_k-k}z_{i_k}^{-1})}{\theta(x_2t^{i_k-k}x_1t^{k-1})\theta(x_2t^{i_k-k}x_1^{-1}t^{-(k-1)})}\\ &\times\prod_{l=1}^{n-r}\frac{\theta(x_1t^{j_l-l}z_{j_l})\theta(x_1t^{j_l-l}z_{j_l}^{-1})}{\theta(x_1t^{j_l-l}x_2t^{l-1})\theta(x_1t^{j_l-l}x_2^{-1}t^{-(l-1)})},\end{aligned}$$

where the summation is taken over all pairs of sequences $1\le i_1<\cdots<i_r\le n$ and $1\le j_1<\cdots<j_{n-r}\le n$ such that $\{i_1,\ldots,i_r\}\cup\{j_1,\ldots,j_{n-r}\}=\{1,2,\ldots,n\}$. In our previous works [9, 10], these functions $E_{(r,n-r)}(x_1,x_2;z)$ $(0\le r\le n)$ are called the BC_n *fundamental invariants*, and are effectively used in an alternative method of evaluation of the BC_n elliptic Selberg integral proposed by van Diejen and Spiridonov [19].

We now return to the connection problem. Another main result of this paper is that the connection coefficient c_μ in (1.7) exactly coincides with our elliptic Lagrange interpolation function $E_\mu(x;z)$ for each $\mu\in Z_{s,n}$.

Theorem 1.5 (Connection formula) *Suppose that $\varphi(z)\in\mathcal{O}((\mathbb{C}^*)^n)$ is W_n-invariant. Then*

$$\langle\!\langle\varphi,z\rangle\!\rangle=\sum_{\mu\in Z_{s,n}}\langle\!\langle\varphi,x_\mu\rangle\!\rangle E_\mu(x;z),\tag{1.12}$$

where the connection coefficients $E_\mu(x;z)$ are explicitly written as (1.10)*.*

We call this connection formula the *generalized Sears–Slater transformation*. In fact, the connection formula (1.12) of the case $n=1$ is given by

$$\langle\!\langle\varphi,z\rangle\!\rangle=\sum_{i=1}^{s}\langle\!\langle\varphi,x_i\rangle\!\rangle\prod_{\substack{1\le j\le s\\ j\ne i}}\frac{\theta(x_jz)\theta(x_j/z)}{\theta(x_ix_j)\theta(x_j/x_i)},$$

which exactly coincides with the transformation formula (1.2) if $\varphi(z) \equiv 1$ and $r = s + 2$. See [11] for details about the correspondence between them.

This paper is organized as follows. We first provide in Section 2 a proof of Theorem 1.3 based on an explicit construction of the elliptic Lagrange interpolation functions by means of a kernel function as in [13]. Section 3 is devoted to proving the explicit formula of Theorem 1.4 for our interpolation functions. In Section 4 we investigate the transition coefficients between two sets of the interpolation functions. In particular we propose an explicit formula for the determinant of the transition matrix. Using properties of the elliptic Lagrange interpolation functions, we complete in Section 5 the proof of the connection formula of Theorem 1.5. The determinant formula of Theorem 1.2 is also obtained as a corollary of Theorem 1.5.

Lastly we remark that another type of BC_n Jackson integral is studied in [8]. In that case, the determinant formula [8, Theorem 1.7] corresponding to Theorem 1.2 is regarded as a generalization of Gustafson's C_n sum. The corresponding connection formula [8, Theorem 1.1] is much simpler than Theorem 1.5 of this paper.

2 Construction of the BC_n interpolation functions

In this section we give a proof of Theorem 1.3.

Throughout this paper we use the symbol

$$e(a;b) = a^{-1}\theta(ab)\theta(ab^{-1}) \qquad (a, b \in \mathbb{C}^*),$$

where $\theta(u) = (u)_\infty(q/u)_\infty$. Since $\theta(a) = \theta(qa^{-1})$ and $\theta(qa) = -a^{-1}\theta(a)$, this symbol satisfies

$$e(a^{-1};b) = e(a;b), \quad e(a;b) = -e(b;a), \quad e(a;a) = 0 \quad \text{and} \quad e(qa;b) = (qa^2)^{-1}e(a;b).$$

Fixing a generic parameter $t \in \mathbb{C}^*$, we also introduce the notation of *t-shifted factorials*

$$e(a;b)_r = e(a;b)e(at;b)\cdots e(at^{r-1};b) \quad (r = 0, 1, 2, \ldots)$$

associated with the symbol $e(a;b)$.

For two sets of variables $z \in (\mathbb{C}^*)^n$ and $y \in (\mathbb{C}^*)^{s-1}$, we consider the *dual Cauchy kernel*

$$\Psi(z;y) = \prod_{i=1}^{n}\prod_{j=1}^{s-1} e(z_i;y_j).$$

Note that $\Psi(z,y)$ is a holomorphic function on $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^{s-1}$, and satisfies

$$\Psi(z;y) \in H^z_{s-1,n} \quad \text{and} \quad \Psi(z;y) \in H^y_{n,s-1},$$

with superscripts indicating the variables. For each multi-index $\mu \in Z_{s,n}$, we define a function $F_\mu(x;y)$ of $y \in (\mathbb{C}^*)^{s-1}$ with parameters $x \in (\mathbb{C}^*)^s$ by

$$F_\mu(x;y) = \Psi(x_\mu;y) = \prod_{i=1}^{s}\prod_{j=1}^{s-1} e(x_i;y_j)_{\mu_i} \quad (x \in (\mathbb{C}^*)^s,\ y \in (\mathbb{C}^*)^{s-1}), \tag{2.1}$$

where $x_\mu \in (\mathbb{C}^*)^n$ is specified by (1.4). These functions $F_\mu(x;y)$ ($\mu \in Z_{s,n}$) are W_{s-1}-invariant with respect to y, and satisfy $T_{y_i}F_\mu(x;y) = (qy_i^2)^{-n}F_\mu(x;y)$ ($i = 1,\dots,s-1$), namely, $F_\mu(x;y) \in H^y_{n,s-1}$. On the other hand, for each $x \in (\mathbb{C}^*)^s$ and $\nu = (\nu_1,\nu_2,\dots,\nu_s) \in Z_{s,n}$, we specify the point $\eta_\nu(x)$ in $(\mathbb{C}^*)^{s-1}$ by

$$\eta_\nu(x) = (x_1t^{\nu_1}, x_2t^{\nu_2}, \dots, x_{s-1}t^{\nu_{s-1}}) \in (\mathbb{C}^*)^{s-1}.$$

Note that the point $\eta_\nu(x)$ has no coordinate corresponding to the sth index ν_s of $\nu \in Z_{s,n}$, while $\eta_\nu(x)$ is defined injectively from $\nu \in Z_{s,n}$ if x is generic. In fact, if we set

$$L_{s,n} = \{(\nu_1,\nu_2,\dots,\nu_{s-1}) \in \mathbb{N}^{s-1}\,;\, \nu_1+\nu_2+\ldots+\nu_{s-1} \le n\},$$

then the map $(\nu_1,\nu_2,\dots,\nu_{s-1},\nu_s) \mapsto (\nu_1,\nu_2,\dots,\nu_{s-1})$ defines a bijection $Z_{s,n} \to L_{s,n}$.

Lemma 2.1 (Triangularity) *For each $\mu,\nu \in Z_{s,n}$, $F_\mu(x;\eta_\nu(x)) = 0$ unless $\mu_i \le \nu_i$ ($i = 1,\dots,s-1$). In particular, $F_\mu(x;\eta_\nu(x)) = 0$ for $\mu \succ \nu$. Moreover, if $x \in (\mathbb{C}^*)^s$ is generic, then $F_\mu(x;\eta_\mu(x)) \neq 0$ for all $\mu \in Z_{s,n}$.*

This lemma implies that the matrix $F = \big(F_\mu(x;\eta_\nu(x))\big)_{\mu,\nu\in Z}$ is upper triangular, and also invertible if $x \in \mathbb{C}^s$ is generic.

Proof. If there exists $j \in \{1,2,\dots,s-1\}$ such that $\nu_j < \mu_j$, then $F_\mu(x;\eta_\nu(x)) = 0$. In fact, in the expression

$$F_\mu(x;\eta_\nu(x)) = \prod_{i=1}^{s}\prod_{j=1}^{s-1} e(x_i;x_jt^{\nu_j})_{\mu_i},$$

the function $e(x_j;x_jt^{\nu_j})_{\mu_j}$ has the factor $\theta(t^{-\nu_j})\theta(t^{-\nu_j+1})\cdots\theta(t^{-\nu_j+(\mu_j-1)}) = 0$ if $\nu_j < \mu_j$. If $\nu \prec \mu$, then $\nu_i < \mu_i$ for some $i \in \{1,2,\dots,s-1\}$ by definition, and hence we obtain $F_\mu(x;\eta_\nu(x)) = 0$ if $\nu \prec \mu$. For $\nu = \mu$,

$$\begin{aligned}F_\mu(x;\eta_\mu(x)) = \prod_{i=1}^{s}\prod_{j=1}^{s-1} &\frac{\theta(x_ix_jt^{\mu_j})}{x_i}\frac{\theta(x_ix_jt^{\mu_j+1})}{x_it}\cdots\frac{\theta(x_ix_jt^{\mu_j+(\mu_i-1)})}{x_it^{\mu_i-1}}\\ &\times\theta(x_ix_j^{-1}t^{-\mu_j})\theta(x_ix_j^{-1}t^{-\mu_j+1})\cdots\theta(x_ix_j^{-1}t^{-\mu_j+(\mu_i-1)})\end{aligned}$$

does not vanish if we impose an appropriate genericity condition on $x \in (\mathbb{C}^*)^s$. □

Lemma 2.2 *The set $\{F_\mu(x;y)\,;\,\mu \in Z_{s,n}\}$ is a basis of the $\mathbb{C}$-linear space $H^y_{n,s-1}$, provided that $x \in (\mathbb{C}^*)^s$ is generic.*

Proof. Since the dimension of $H^y_{n,s-1}$ is $\binom{n+s-1}{n}$, it suffices to show $\{F_\mu(x;y)\,;\,\mu \in Z_{s,n}\}$ is linearly independent. It is confirmed from the fact

$$\det F = \det\big(F_\mu(x;\eta_\nu(x))\big)_{\mu,\nu\in Z} = \prod_{\mu\in Z_{s,n}} F_\mu(x;\eta_\mu(x)) \neq 0,$$

which is a consequence of Lemma 2.1. □

Proof of Theorem 1.3. We fix a generic point x in $(\mathbb{C}^*)^s$. Since $\Psi(z;y)$ as a function of y is in $H^y_{n,s-1}$ by definition, using Lemma 2.2, it is expressed as a linear combination of $F_\mu(x;y)$ $(\mu\in Z_{s,n})$, i.e.

$$\Psi(z;y)=\prod_{i=1}^{n}\prod_{j=1}^{s-1}e(z_i;y_j)=\sum_{\mu\in Z_{s,n}}f_\mu(z)F_\mu(x;y), \tag{2.2}$$

where $f_\lambda(z)$ are coefficients independent of y. Substituting $y=\eta_\nu(x)$ in this formula, we have

$$\Psi(z;\eta_\nu(x))=\sum_{\mu\in Z_{s,n}}f_\mu(z)F_\mu(x;\eta_\nu(x))\qquad(\nu\in Z_{s,n}).$$

In what follows, we denote by $G=(G_{\mu,\nu}(x))_{\mu,\nu\in Z}$ the inverse matrix of $F=(F_\mu(x;\eta_\nu(x)))_{\mu,\nu\in Z}$. Then we obtain

$$f_\lambda(z)=\sum_{\nu\in Z_{s,n}}\Psi(z;\eta_\nu(x))G_{\nu\lambda}(x)\qquad(\lambda\in Z_{s,n}), \tag{2.3}$$

which implies $f_\lambda(z)\in H^z_{s-1,n}$ since $\Psi(z;\eta_\nu(x))\in H^z_{s-1,n}$. Setting $z=x_\mu$ for each $\mu\in Z_{s,n}$ as in (1.4), we obtain

$$f_\lambda(x_\mu)=\sum_{\nu\in Z_{s,n}}\Psi(x_\mu,\eta_\nu(x))G_{\nu\lambda}(x)=\sum_{\nu\in Z_{s,n}}F_\mu(x;\eta_\nu(x))G_{\nu\lambda}(x)=\delta_{\lambda\mu}\quad(\lambda,\mu\in Z_{s,n}). \tag{2.4}$$

These functions $f_\lambda(z)\in H^z_{s-1,n}$ $(\lambda\in Z_{s,n})$ are exactly what we wanted to construct. □

Denoting the functions $f_\lambda(z)$ by $E_\lambda(x;z)$, we call them the *elliptic Lagrange interpolation functions* of type BC_n. We restate (2.2) as corollary, which indicates the duality between $E_\lambda(x;z)$ and $F_\lambda(x;y)$.

Corollary 2.3 (Duality)

$$\Psi(z;y)=\prod_{i=1}^{n}\prod_{j=1}^{s-1}e(z_i,y_j)=\sum_{\lambda\in Z_{s,n}}E_\lambda(x;z)F_\lambda(x;y). \tag{2.5}$$

We will use this formula again in the succeeding sections.

From (2.3) we have

$$E_\lambda(x;z)=\sum_{\substack{\mu\in Z_{s,n}\\ \mu\preceq\lambda}}\Psi(z;\eta_\mu(x))G_{\mu\lambda}(x), \tag{2.6}$$

which leads us to another explicit expression of $E_\lambda(x;z)$ different from (1.10) in Theorem 1.4. Recall that, for an invertible upper triangular matrix $A=(a_{ij})_{i,j=1}^N$ with $a_{ij}=0$ $(i>j)$, the entries of its inverse are given by

$$(A^{-1})_{ij}=\sum_{r=0}^{N-1}(-1)^r\sum_{i=k_0<k_1<\cdots<k_r=j}\frac{a_{k_0k_1}a_{k_1k_2}\cdots a_{k_{r-1}k_r}}{a_{k_0k_0}a_{k_1k_1}a_{k_2k_2}\cdots a_{k_rk_r}}\quad(i\le j). \tag{2.7}$$

In fact, setting $D=\mathrm{diag}(a_{11},\ldots,a_{NN})$ and $B=A-D$, from $A=D(I+D^{-1}B)$ we obtain

$$\begin{aligned}A^{-1}=(I+D^{-1}B)^{-1}D^{-1}&=\sum_{r=0}^{N-1}(-1)^r(D^{-1}B)^rD^{-1}\\&=\sum_{r=0}^{N-1}(-1)^rD^{-1}BD^{-1}\cdots D^{-1}BD^{-1},\end{aligned}$$

whose (i,j)-components are given by (2.7).

Corollary 2.4 *The elliptic Lagrange interpolation functions $E_\lambda(x;z)$ are expressed explicitly as*

$$E_\lambda(x;z)=\sum_{\substack{\mu\in Z_{s,n}\\ \mu\preceq\lambda}}\left(\sum_{r\geq 0}(-1)^r\sum_{\mu=\nu^{(0)}\prec\cdots\prec\nu^{(r)}=\lambda}\frac{\prod_{k=1}^r\prod_{i=1}^s\prod_{j=1}^{s-1}e(x_i;x_jt^{\nu_j^{(k)}})_{\nu_i^{(k-1)}}}{\prod_{k=0}^r\prod_{i=1}^s\prod_{j=1}^{s-1}e(x_i;x_jt^{\nu_j^{(k)}})_{\nu_i^{(k)}}}\right)\prod_{i=1}^n\prod_{j=1}^{s-1}e(z_i;x_jt^{\mu_j}).$$

Proof. In (2.6), $G=\big(G_{\mu\nu}(x)\big)_{\mu,\nu\in Z}$ is the inverse matrix of the upper triangular matrix $F=\big(F_\mu(x;\eta_\nu(x))\big)_{\mu,\nu\in Z}$. By (2.7) we have the expression

$$\begin{aligned}G_{\mu\lambda}(x)&=\sum_{r\geq 0}(-1)^r\sum_{\mu=\nu^{(0)}\prec\cdots\prec\nu^{(r)}=\lambda}\frac{\prod_{k=1}^rF_{\nu^{(k-1)}}(x;\eta_{\nu^{(k)}})}{\prod_{k=0}^rF_{\nu^{(k)}}(x;\eta_{\nu^{(k)}})}\\&=\sum_{r\geq 0}(-1)^r\sum_{\mu=\nu^{(0)}\prec\cdots\prec\nu^{(r)}=\lambda}\frac{\prod_{k=1}^r\prod_{i=1}^s\prod_{j=1}^{s-1}e(x_i;x_jt^{\nu_j^{(k)}})_{\nu_i^{(k-1)}}}{\prod_{k=0}^r\prod_{i=1}^s\prod_{j=1}^{s-1}e(x_i;x_jt^{\nu_j^{(k)}})_{\nu_i^{(k)}}}.\end{aligned}\tag{2.8}$$

By definition we also have

$$\Psi(z;\eta_\mu(x))=\prod_{i=1}^n\prod_{j=1}^{s-1}e(z_i;x_jt^{\mu_j}).\tag{2.9}$$

Putting (2.8) and (2.9) on (2.6) we have the expression in Corollary. □

Remark. The expression of $E_\lambda(x;z)$ in Corollary 2.4 is much more complex than (1.10) in Theorem 1.4. Actually, it often becomes a huge sum even in the case where $E_\lambda(x;z)$ can be written simply in the form of product like (3.10).

3 Explicit expression for $E_\lambda(x;z)$

In this section we give a proof of Theorem 1.4. The main part of the proof is due to the repeated use of the recurrence relation for our interpolation functions. We first mention the explicit form of the interpolation functions of the case $n=1$ as the initial step of the recursive process.

Lemma 3.1 *For $x=(x_1,x_2,\ldots,x_s)\in(\mathbb{C}^*)^s$ and $z\in\mathbb{C}^*$ the interpolation functions of the case $n=1$ are expressed explicitly as*

$$E_{\epsilon_i}(x;z)=\prod_{\substack{1\leq j\leq s\\ j\neq i}}\frac{e(z;x_j)}{e(x_i;x_j)}\tag{3.1}$$

for $i=1,2,\ldots,s$, where $\epsilon_i=(0,\ldots,0,\overset{i}{\breve{1}},0,\ldots,0)\in Z_{s,1}=\{\epsilon_1,\epsilon_2,\ldots,\epsilon_s\}$.

Proof. Since $x_{\epsilon_i} = x_i$, it is immediately verified that the functions of the right-hand side of (3.1) satisfy the conditions $E_{\epsilon_i}(x;z) \in H^z_{s-1,1}$ and $E_{\epsilon_i}(x;x_{\epsilon_j}) = \delta_{ij}$. Such functions are determined uniquely by Theorem 1.3. □

Next we state the recursion formula for the interpolation functions.

Lemma 3.2 (Recursion formula) *Suppose that $n = m + l$. For $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ written as $z = (z', z'')$ where $z' = (z_1, \dots, z_m) \in (\mathbb{C}^*)^m$ and $z'' = (z_{m+1}, \dots, z_n) \in (\mathbb{C}^*)^l$, the function $E_\lambda(x;z)$ ($\lambda \in Z_{s,n}$) is expressed as*

$$E_\lambda(x;z) = \sum_{\substack{\mu \in Z_{s,m}, \nu \in Z_{s,l} \\ \mu+\nu=\lambda}} E_\mu(x;z') E_\nu(xt^\mu; z''), \tag{3.2}$$

where $xt^\mu = (x_1 t^{\mu_1}, x_2 t^{\mu_2}, \dots, x_s t^{\mu_s})$ for $x = (x_1, x_2, \dots, x_s) \in (\mathbb{C}^*)^s$.

Proof. From the definition (2.1) of $F_\mu(x;y)$, it follows that

$$F_\mu(x;y) F_\nu(xt^\mu; y) = F_{\mu+\nu}(x;y).$$

From this fact and (2.5) we have

$$\begin{aligned}
\Psi(z;y) &= \Psi(z';y)\Psi(z'';y) \\
&= \sum_{\mu \in Z_{s,m}} E_\mu(x;z') F_\mu(x;y) \Psi(z'';y) \\
&= \sum_{\mu \in Z_{s,m}} E_\mu(x;z') F_\mu(x;y) \Big(\sum_{\nu \in Z_{s,l}} E_\nu(xt^\mu; z'') F_\nu(xt^\mu; y) \Big) \\
&= \sum_{\mu \in Z_{s,m}} \sum_{\nu \in Z_{s,l}} E_\mu(x;z') E_\nu(xt^\mu; z'') F_\mu(x;y) F_\nu(xt^\mu; y) \\
&= \sum_{\mu \in Z_{s,m}} \sum_{\nu \in Z_{s,l}} E_\mu(x;z') E_\nu(xt^\mu; z'') F_{\mu+\nu}(x;y) \\
&= \sum_{\lambda \in Z_{s,n}} \Bigg[\sum_{\substack{\mu \in Z_{s,m}, \nu \in Z_{s,l} \\ \mu+\nu=\lambda}} E_\mu(x;z') E_\nu(xt^\mu; z'') \Bigg] F_\lambda(x;y).
\end{aligned} \tag{3.3}$$

Comparing (3.3) with (2.5), we obtain the expression (3.2) in Lemma 3.2. □

Remark. The special cases $m = 1$ or $l = 1$ of the recursion formula in Lemma 3.2 indicate that

$$E_\lambda(x;z) = \sum_{i=1}^{s} E_{\epsilon_i}(x;z_1) E_{\lambda-\epsilon_i}(xt^{\epsilon_i}; z_2, \dots, z_n) \tag{3.4}$$

or

$$E_\lambda(x;z) = \sum_{i=1}^{s} E_{\lambda-\epsilon_i}(x;z_1, \dots, z_{n-1}) E_{\epsilon_i}(xt^{\lambda-\epsilon_i}; z_n), \tag{3.5}$$

where we regard $E_{\lambda-\epsilon_i}(x;z_1, \dots, z_{n-1}) = 0$ if $\lambda - \epsilon_i \notin Z_{s,n-1}$.

By the repeated use of (3.4) or (3.5) we have the following expression.

Corollary 3.3

$$E_\lambda(x;z) = \sum_{\substack{(i_1,\ldots,i_n)\in\{1,\ldots,s\}^n \\ \epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda}} E_{\epsilon_{i_1}}(x;z_1)E_{\epsilon_{i_2}}(xt^{\epsilon_{i_1}};z_2)E_{\epsilon_{i_3}}(xt^{\epsilon_{i_1}+\epsilon_{i_2}};z_3)\cdots E_{\epsilon_{i_n}}(xt^{\epsilon_{i_1}+\cdots+\epsilon_{i_{n-1}}};z_n).$$

Rewriting Corollary 3.3, we obtain the explicit formula (1.10) for $E_\lambda(x;z)$ as presented in Theorem 1.4.

Theorem 3.4 *The interpolation functions $E_\lambda(x;z)$ are expressed explicitly as*

$$E_\lambda(x;z) = \sum_{\substack{(i_1,\ldots,i_n)\in\{1,\ldots,s\}^n \\ \epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda}} \prod_{k=1}^{n} \prod_{\substack{1\le j\le s \\ j\ne i_k}} \frac{e(z_k\,;x_jt^{\lambda_j^{(k-1)}})}{e(x_{i_k}t^{\lambda_{i_k}^{(k-1)}}\,;x_jt^{\lambda_j^{(k-1)}})}, \tag{3.6}$$

where $\lambda_i^{(k)} = |\{l\in\{1,\ldots,k\}\,;\,i_l=i\}|$. Equivalently $E_\lambda(x;z)$ is also written as (1.10) *in Theorem 1.4, i.e.,*

$$E_\lambda(x;z) = \sum_{\substack{K_1\sqcup\cdots\sqcup K_s \\ =\{1,2,\ldots,n\}}} \prod_{i=1}^{s} \prod_{k\in K_i} \prod_{\substack{1\le j\le s \\ j\ne i}} \frac{e(z_k\,;x_jt^{\lambda_j^{(k-1)}})}{e(x_it^{\lambda_i^{(k-1)}}\,;x_jt^{\lambda_j^{(k-1)}})}, \tag{3.7}$$

where $\lambda_i^{(k)} = |K_i\cap\{1,2,\ldots,k\}|$ and the summation is taken over all index sets K_i $(i=1,2,\ldots,s)$ satisfying $|K_i|=\lambda_i$ and $K_1\sqcup\cdots\sqcup K_s=\{1,2,\ldots,n\}$.

Proof. For $(i_1,\ldots,i_n)\in\{1,\ldots,s\}^n$ satisfying $\epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda$, we set $\lambda^{(k)}=\epsilon_{i_1}+\cdots+\epsilon_{i_k}$ for $k=0,1,\ldots,n$. Then by definition $\lambda^{(k)}=(\lambda_1^{(k)},\ldots,\lambda_s^{(k)})\in Z_{s,k}$ is expressed by

$$\lambda_i^{(k)} = |\{l\in\{1,\ldots,k\}\,;\,i_l=i\}| \quad (i=1,\ldots,s).$$

From Corollary 3.3, we therefore obtain

$$E_\lambda(x;z) = \sum_{\substack{(i_1,\ldots,i_n)\in\{1,\ldots,s\}^n \\ \epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda}} \prod_{k=1}^{n} E_{\epsilon_{i_k}}(xt^{\lambda^{(k-1)}};z_k), \tag{3.8}$$

which coincides with (3.6) using (3.1).

Next we explain the latter part of the theorem. Let K_i be sets of indices specified by $K_i=\{l\in\{1,\ldots,n\}\,;\,i_l=i\}$, where $(i_1,\ldots,i_n)\in\{1,\ldots,s\}^n$ and $\epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda$. Then $\lambda_i^{(k)}$ is written as

$$\lambda_i^{(k)} = |K_i\cap\{1,2,\ldots,k\}|.$$

In particular, we have $\lambda_i=\lambda_i^{(n)}=|K_i|$. Thus K_i $(i=1,2,\ldots,s)$ satisfy $K_1\sqcup\cdots\sqcup K_s=\{1,2,\ldots,n\}$. Since $i_k=i$ if and only if $k\in K_i$, the expression (3.8) is rewritten as

$$E_\lambda(x;z) = \sum_{\substack{K_1\sqcup\cdots\sqcup K_s \\ =\{1,2,\ldots,n\}}} \prod_{i=1}^{s} \prod_{k\in K_i} E_{\epsilon_i}(xt^{\lambda^{(k-1)}};z_k) \tag{3.9}$$

where the summation is taken over all index sets K_i satisfying $|K_i|=\lambda_i$ and $K_1\sqcup\cdots\sqcup K_s=\{1,2,\ldots,n\}$. Therefore (3.9) coincides with (3.7) using (3.1). □

We remark that the interpolation functions of the special cases $\lambda=n\epsilon_i\in Z_{s,n}$ have simple factorized forms; this fact will be used in the succeeding section.

Corollary 3.5 *For $n\epsilon_i \in Z_{s,n}$ $(i = 1, \ldots, s)$, one has*

$$E_{n\epsilon_i}(x; z) = \prod_{\substack{1\le j\le s\\ j\ne i}} \frac{e(z_1; x_j) \cdots e(z_n; x_j)}{e(x_i; x_j)_n}. \tag{3.10}$$

Proof. If we put $\lambda = n\epsilon_i$ in the formula of Corollary 3.3, then the right-hand side reduces to a single term with $(i_1, i_2, \ldots, i_n) = (i, i, \ldots, i)$. Therefore, using (3.1) we obtain

$$\begin{aligned} E_{n\epsilon_i}(x; z) &= E_{\epsilon_i}(x; z_1)E_{\epsilon_i}(xt^{\epsilon_i}; z_2)E_{\epsilon_i}(xt^{2\epsilon_i}; z_3) \cdots E_{\epsilon_i}(xt^{(n-1)\epsilon_i}; z_n) \\ &= \prod_{\substack{1\le j\le s\\ j\ne i}} \frac{e(z_1; x_j)}{e(x_i; x_j)} \frac{e(z_2; x_j)}{e(x_it; x_j)} \frac{e(z_3; x_j)}{e(x_it^2; x_j)} \cdots \frac{e(z_n; x_j)}{e(x_it^{n-1}; x_j)}, \end{aligned}$$

which coincides with (3.10). □

4 Transition coefficients for the interpolation functions

In this section we discuss the transition coefficients between two sets of interpolation functions with different parameters.

For generic $x, y \in (\mathbb{C}^*)^s$, the interpolation functions $E_\mu(x; z) \in H^z_{s-1,n}$ as functions of $z \in (\mathbb{C}^*)^n$ are expanded in terms of $E_\nu(y; z)$ $(\nu \in Z_{s,n})$, i.e.,

$$E_\mu(x; z) = \sum_{\nu\in Z_{s,n}} C_{\mu\nu}(x; y)E_\nu(y; z), \tag{4.1}$$

where the coefficients $C_{\mu\nu}(x; y)$ are independent of z. From the property (1.9) of the interpolation functions, we immediately see that $C_{\mu\nu}(x; y)$ is expressed by the special value of $E_\mu(x; z)$ as

$$C_{\mu\nu}(x; y) = E_\mu(x; y_\nu) \quad (\mu, \nu \in Z_{s,n}).$$

For $x, y \in (\mathbb{C}^*)^s$, we denote the *transition matrix* from $(E_\lambda(x; z))_{\lambda\in Z_{s,n}}$ to $(E_\lambda(y; z))_{\lambda\in Z_{s,n}}$ by

$$E(x; y) = \Big(E_\mu(x; y_\nu)\Big)_{\mu,\nu\in Z_{s,n}},$$

where the rows and the columns are arranged in the total order $\prec$ of $Z_{s,n}$. By definition, for generic $x, y, w \in (\mathbb{C}^*)^s$ we have

$$E(x; y) = E(x; w)E(w; y), \tag{4.2}$$

in particular

$$E(x; x) = I \quad \text{and} \quad E(y; x) = E(x; y)^{-1}. \tag{4.3}$$

Theorem 4.1 *For generic $x, y \in (\mathbb{C}^*)^s$ the determinant of the transition matrix $E(x; y)$ is given explicitly by*

$$\det E(x; y) = \prod_{k=1}^{n} \left[\prod_{r=0}^{n-k} \prod_{1\le i<j\le s} \frac{e(y_it^r; y_jt^{(n-k)-r})}{e(x_it^r; x_jt^{(n-k)-r})} \right]^{\binom{s+k-3}{k-1}}, \tag{4.4}$$

or equivalently by

$$\det E(x;y) = \prod_{k=1}^{n}\Bigg[\prod_{r=0}^{n-k}\prod_{1\le i<j\le s}\frac{x_i\theta(t^{2r-(n-k)}y_iy_j^{-1})\theta(t^{n-k}y_iy_j)}{y_i\theta(t^{2r-(n-k)}x_ix_j^{-1})\theta(t^{n-k}x_ix_j)}\Bigg]^{\binom{s+k-3}{k-1}}. \tag{4.5}$$

The goal of this section is to prove the above theorem. For this purpose we first investigate a special case.

Lemma 4.2 *For $x, y \in (\mathbb{C}^*)^s$ suppose that $y_i = x_i$ $(i = 1, 2, \ldots, s-1)$, i.e. $y = (x_1, \ldots, x_{s-1}, y_s)$. For $\alpha, \beta \in Z_{s,n}$ if there exists $i \in \{1, 2, \ldots, s-1\}$ such that $\alpha_i < \beta_i$, then $E_\alpha(x; y_\beta) = 0$. In particular, $E(x, y)$ is a lower triangular matrix with the diagonal entries*

$$E_\alpha(x;y_\alpha) = \prod_{i=1}^{s-1}\frac{e(y_s;x_it^{\alpha_i})_{\alpha_s}}{e(x_s;x_it^{\alpha_i})_{\alpha_s}}.$$

Moreover the determinant of $E(x,y)$ of the case $y = (x_1, \ldots, x_{s-1}, y_s)$ is expressed as

$$\det E(x,y) = \prod_{\alpha\in Z_{s,n}}\prod_{i=1}^{s-1}\frac{e(y_s;x_it^{\alpha_i})_{\alpha_s}}{e(x_s;x_it^{\alpha_i})_{\alpha_s}} = \prod_{k=1}^{n}\Bigg[\prod_{r=0}^{n-k}\prod_{i=1}^{s-1}\frac{e(x_it^r;y_st^{(n-k)-r})}{e(x_it^r;x_st^{(n-k)-r})}\Bigg]^{\binom{s+k-3}{k-1}}. \tag{4.6}$$

Proof. If $\beta_s = 0$ for $\beta \in Z_{s,n}$, then $C_{\alpha\beta}(x,y) = E_\alpha(x;y_\beta) = \delta_{\alpha\beta}$ by the definition (4.1). If $\beta_s \neq 0$ for $\beta \in Z_{s,n}$, we apply Lemma 3.2 with $m = \beta_1 + \cdots + \beta_{s-1}$, $l = \beta_s$ to obtain

$$\begin{aligned} C_{\alpha\beta}(x;y) = E_\alpha(x;y_\beta) &= \sum_{\substack{\mu\in Z_{s,m},\nu\in Z_{s,l}\\ \mu+\nu=\alpha}} E_\mu(x;y_{\beta'})E_\nu(xt^\mu;y_{\beta''}) \\ &= \sum_{\substack{\mu\in Z_{s,m},\nu\in Z_{s,l}\\ \mu+\nu=\alpha}} E_\mu(x;x_{\beta'})E_\nu(xt^\mu;y_{\beta''}), \end{aligned} \tag{4.7}$$

where $\beta' = (\beta_1, \ldots, \beta_{s-1}, 0) \in Z_{s,m}$, and $\beta'' = (0, \ldots, 0, \beta_s) \in Z_{s,l}$ so that $y_{\beta'} = x_{\beta'}$. Note that

$$x_{\beta'} = (\underbrace{x_1, x_1t, \ldots, x_1t^{\beta_1-1}}_{\beta_1}, \underbrace{x_2, x_2t, \ldots, x_2t^{\beta_2-1}}_{\beta_2}, \ldots, \underbrace{x_{s-1}, x_{s-1}t, \ldots, x_{s-1}t^{\beta_{s-1}-1}}_{\beta_{s-1}}) \in (\mathbb{C}^*)^m,$$

$$y_{\beta''} = (y_s, y_st, \ldots, y_st^{\beta_s-1}) \in (\mathbb{C}^*)^l.$$

By the property of the interpolation functions, we have $E_\mu(x;x_{\beta'}) = \delta_{\mu\beta'}$ for $\mu \in Z_{s,m}$. From (4.7) $C_{\alpha\beta}(x;y)$ is written as

$$C_{\alpha\beta}(x;y) = \sum_{\substack{\mu\in Z_{s,m},\nu\in Z_{s,l}\\ \mu+\nu=\alpha}} \delta_{\mu\beta'}E_\nu(xt^\mu;y_{\beta''}) = E_{\alpha-\beta'}(xt^{\beta'};y_{\beta''}), \tag{4.8}$$

where $\alpha - \beta' = (\alpha_1 - \beta_1, \ldots, \alpha_{s-1} - \beta_{s-1}, \alpha_s) \in Z_{s,l}$ and $xt^{\beta'} = (x_1t^{\beta_1}, x_2t^{\beta_2}, \ldots, x_{s-1}t^{\beta_{s-1}}, x_s) \in (\mathbb{C}^*)^s$. If there exists $i \in \{1, 2, \ldots, s-1\}$ such that $\alpha_i < \beta_i$, i.e., $\alpha - \beta' \notin Z_{s,l}$, then $C_{\alpha\beta}(x;y) = 0$. In particular, if $\alpha \prec \beta$, then $C_{\alpha\beta}(x;y) = 0$, which indicates the matrix $E(x,y)$ of the case $y = (x_1, \cdots, x_{s-1}, y_s)$ is lower triangular. On the other hand, if $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2, \ldots, \alpha_{s-1} \geq \beta_{s-1}$

(and $\alpha_s = n - (\alpha_1 + \cdots + \alpha_{s-1}) \le n - (\beta_1 + \cdots + \beta_{s-1}) = \beta_s$), $C_{\alpha\beta}(x;y)$ is written as (4.8). In particular, if $\alpha = \beta$, then

$$C_{\beta\beta}(x;y) = E_{\beta''}(xt^{\beta'}; y_{\beta''}). \tag{4.9}$$

Since $\beta'' = \beta_s\epsilon_s$, from (3.10) in Corollary 3.5 we have

$$E_{\beta''}(xt^{\beta'}; z_{m+1}, \ldots, z_n) = \prod_{i=1}^{s-1} \frac{e(z_{m+1}; x_i t^{\beta_i}) \cdots e(z_n; x_i t^{\beta_i})}{e(x_s; x_i t^{\beta_i})_{\beta_s}}. \tag{4.10}$$

From (4.9) and (4.10) we therefore obtain

$$C_{\beta\beta}(x;y) = E_\beta(x; y_\beta) = \prod_{i=1}^{s-1} \frac{e(y_s; x_i t^{\beta_i})_{\beta_s}}{e(x_s; x_i t^{\beta_i})_{\beta_s}}.$$

Lastly we derive (4.6). Since the matrix $E(x;y)$ is lower triangular, its determinant is calculated as

$$\begin{aligned}\det\Big(E_\alpha(x;y_\beta)\Big)_{\alpha,\beta\in Z_{s,n}} &= \prod_{\alpha\in Z_{s,n}} \prod_{i=1}^{s-1} \frac{e(y_s; x_i t^{\alpha_i})_{\alpha_s}}{e(x_s; x_i t^{\alpha_i})_{\alpha_s}} \\ = \prod_{i=1}^{s-1} \prod_{\alpha\in Z_{s,n}} \prod_{l=1}^{\alpha_s} \frac{e(y_s t^{l-1}; x_i t^{\alpha_i})}{e(x_s t^{l-1}; x_i t^{\alpha_i})} &= \prod_{i=1}^{s-1} \prod_{r=0}^{n-1} \prod_{l=1}^{n-r} \prod_{\substack{\alpha\in Z_{s,n} \\ \alpha_i = r,\, \alpha_s \ge l}} \frac{e(y_s t^{l-1}; x_i t^r)}{e(x_s t^{l-1}; x_i t^r)}.\end{aligned} \tag{4.11}$$

Here we count the number of $\alpha \in Z_{s,n}$ such that $\alpha_i = r$ and $\alpha_s \ge l$. Note that for $0 \le r + k \le n$, $|\{\alpha \in Z_{s,n}\,;\, \alpha_i = r, \alpha_s = k\}| = |Z_{s-2,n-r-k}| = \binom{n-r-k+s-3}{s-3}$. Hence for $r + l \le n$, we have $|\{\alpha \in Z_{s,n}\,;\, \alpha_i = r, \alpha_s \ge l\}| = \sum_{k=l}^{n-r} \binom{n-r-k+s-3}{s-3} = \sum_{p=0}^{n-r-l} \binom{p+s-3}{s-3} = \binom{n-r-l+s-2}{s-2}$. Finally we obtain

$$\begin{aligned}\det\Big(E_\alpha(x;y_\beta)\Big)_{\alpha,\beta\in Z_{s,n}} &= \prod_{i=1}^{s-1} \prod_{r=0}^{n-1} \prod_{l=1}^{n-r} \left(\frac{e(y_s t^{l-1}; x_i t^r)}{e(x_s t^{l-1}; x_i t^r)}\right)^{\binom{n-r-l+s-2}{s-2}} \\ = \prod_{i=1}^{s-1} \prod_{r=0}^{n-1} \prod_{k=1}^{n-r} \left(\frac{e(y_s t^{n-r-k}; x_i t^r)}{e(x_s t^{n-r-k}; x_i t^r)}\right)^{\binom{k+s-3}{s-2}} &= \prod_{i=1}^{s-1} \prod_{k=1}^{n} \prod_{r=0}^{n-k} \left(\frac{e(y_s t^{n-r-k}; x_i t^r)}{e(x_s t^{n-r-k}; x_i t^r)}\right)^{\binom{k+s-3}{k-1}},\end{aligned} \tag{4.12}$$

which coincides with (4.6). The proof is now complete. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. We set

$$w^{(i)} = (x_1, \ldots, x_i, y_{i+1}, \ldots, y_s) \in (\mathbb{C}^*)^s$$

for $i = 0, 1, \ldots, s$, which satisty $w^{(s)} = x$ and $w^{(0)} = y$. Since $w^{(s-1)} = (x_1, \ldots, x_{s-1}, y_s)$, Lemma 4.2 indicates that

$$\det E(w^{(s)}; w^{(s-1)}) = \prod_{\alpha\in Z_{s,n}} \prod_{i=1}^{s-1} \frac{e(y_s; x_i t^{\alpha_i})_{\alpha_s}}{e(x_s; x_i t^{\alpha_i})_{\alpha_s}} = \prod_{k=1}^{n} \left[\prod_{r=0}^{n-k} \prod_{i=1}^{s-1} \frac{e(x_i t^r; y_s t^{(n-k)-r})}{e(x_i t^r; x_s t^{(n-k)-r})}\right]^{\binom{s+k-3}{k-1}}.$$

In the same way as Lemma 4.2, for $l = 1, \dots, s$ we have

$$
\begin{aligned}
\det E(w^{(l)}; w^{(l-1)}) &= \prod_{\alpha \in Z_{s,n}} \left[\Big(\prod_{1 \le i < l} \frac{e(y_l; x_i t^{\alpha_i})_{\alpha_l}}{e(x_l; x_i t^{\alpha_i})_{\alpha_l}} \Big) \Big(\prod_{l < j \le s} \frac{e(y_l; y_j t^{\alpha_j})_{\alpha_l}}{e(x_l; y_j t^{\alpha_j})_{\alpha_l}} \Big) \right] \\
&= \prod_{k=1}^{n} \prod_{r=0}^{n-k} \left[\Big(\prod_{1 \le i < l} \frac{e(x_i t^r; y_l t^{(n-k)-r})}{e(x_i t^r; x_l t^{(n-k)-r})} \Big) \Big(\prod_{l < j \le s} \frac{e(y_l t^r; y_j t^{(n-k)-r})}{e(x_l t^r; y_j t^{(n-k)-r})} \Big) \right]^{\binom{s+k-3}{k-1}}
\end{aligned}
\tag{4.13}
$$

exchanging the roles of indices l and s. From the relation (4.2) of transition matrices, we have the decomposition of $E(x; y)$ as

$$
E(x; y) = E(w^{(s)}; w^{(0)}) = E(w^{(s)}; w^{(s-1)}) E(w^{(s-1)}; w^{(s-2)}) \cdots E(w^{(1)}; w^{(0)}). \tag{4.14}
$$

Applying (4.13) to (4.14) we obtain

$$
\det E(x; y) = \prod_{l=1}^{s} \det E(w^{(l-1)}; w^{(l)}) = \prod_{k=1}^{n} \left[\prod_{r=0}^{n-k} \prod_{1 \le i < j \le s} \frac{e(y_i t^r; y_j t^{(n-k)-r})}{e(x_i t^r; x_j t^{(n-k)-r})} \right]^{\binom{s+k-3}{k-1}},
$$

which completes the proof of Theorem 4.1. □

Remark. In the decomposition (4.14), each component $E(w^{(l)}; w^{(l-1)})$ $(l = 1, \dots, s)$ is lower triangular with respect to the partial ordering $\subseteq_l$ of $Z_{s,n}$ defined by $\alpha \subseteq_l \beta \iff \alpha_i \le \beta_i \ (i \ne l)$.

5 Proofs of the main theorems for BC_n Jackson integrals

We conclude this paper by providing with proofs of Theorems 1.2 and 1.5, on the basis of properties of the BC_n elliptic Langrange interpolation functions $E_\lambda(x; z)$ as we established in the previous sections.

Proof of Theorem 1.5. If $\varphi(z)$ is a W_n-invariant holomorphic function on $(\mathbb{C}^*)^n$, then $\langle \varphi, z \rangle$ is a meromorphic function on $(\mathbb{C}^*)^n$ and q-periodic with respect to each variable z_i $(i = 1, \dots, s)$. It is known by [3, Definition 3.8] that the regularization $\langle\!\langle \varphi, z \rangle\!\rangle = \langle \varphi, z \rangle / \Theta(z)$ is a W_n-invariant holomorphic function on $(\mathbb{C}^*)^n$, and belongs to $H^z_{s-1,n}$ due to the quasi-periodicity of $1/\Theta(z)$. This implies that $\langle\!\langle \varphi, z \rangle\!\rangle$ is expressed as a linear combination of our interpolation functions $E_\mu(x; z)$ $(\mu \in Z_{s,n})$, i.e.,

$$
\langle\!\langle \varphi, z \rangle\!\rangle = \sum_{\mu \in Z} d_\mu E_\mu(x; z).
$$

From (1.9), we obtain $d_\nu = \sum_{\mu \in Z} d_\mu E_\mu(x; x_\nu) = \langle\!\langle \varphi, x_\nu \rangle\!\rangle$, which completes the proof of Theorem 1.5. □

Proof of Theorem 1.2. The spacial case $x = a = (a_1, \dots, a_s)$ of Theorem 1.2 was proved in [3, Theorem 1.3], i.e.,

$$
\begin{aligned}
\det \Big(\langle\!\langle \chi_\lambda, a_\mu \rangle\!\rangle \Big)_{\substack{\lambda \in B \\ \mu \in Z}} &= \prod_{k=1}^{n} \left[\Big((1-q) \frac{(q)_\infty (q t^{-(n-k+1)})_\infty}{(q t^{-1})_\infty} \Big)^s \frac{\prod_{1 \le i < j \le 2s+2} (q t^{-(n-k)} a_i^{-1} a_j^{-1})_\infty}{(q t^{-(n+k-2)} a_1^{-1} a_2^{-1} \cdots a_{2s+2}^{-1})_\infty} \right]^{\binom{s+k-2}{k-1}} \\
&\times \prod_{k=1}^{n} \left[\prod_{r=0}^{n-k} \prod_{1 \le i < j \le s} \frac{\theta(t^{2r-(n-k)} a_i a_j^{-1}) \theta(t^{n-k} a_i a_j)}{t^r a_i} \right]^{\binom{s+k-3}{k-1}}.
\end{aligned}
\tag{5.1}
$$

On the other hand, if we put $x = a = (a_1, \ldots, a_s)$ on (1.12) in Theorem 1.5, then we have

$$\langle\!\langle \varphi, z \rangle\!\rangle = \sum_{\mu \in Z} \langle\!\langle \varphi, a_\mu \rangle\!\rangle E_\mu(a; z).$$

In particular, setting $\varphi(z) = \chi_\lambda(z)$ $(\lambda \in B_{s,n})$ and $z = x_\nu$ $(\nu \in Z_{s,n})$ we obtain

$$\Big(\langle\!\langle \chi_\lambda, x_\nu \rangle\!\rangle\Big)_{\substack{\lambda \in B \\ \nu \in Z}} = \Big(\langle\!\langle \chi_\lambda, a_\mu \rangle\!\rangle\Big)_{\substack{\lambda \in B \\ \mu \in Z}} \Big(E_\mu(a; x_\nu)\Big)_{\substack{\mu \in Z \\ \nu \in Z}},$$

so that

$$\det \Big(\langle\!\langle \chi_\lambda, x_\nu \rangle\!\rangle\Big)_{\substack{\lambda \in B \\ \nu \in Z}} = \det \Big(\langle\!\langle \chi_\lambda, a_\mu \rangle\!\rangle\Big)_{\substack{\lambda \in B \\ \mu \in Z}} \det E(a; x).$$

Combining (5.1) and (4.5), we obtain the determinant formula of (1.5). This completes the proof of Theorem 1.2. □

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